ON THE SQUARE ROOT OF THE CENTRE OF THE HECKE ALGEBRA OF TYPE A

ANDREW FRANCIS AND LENNY JONES

ABSTRACT. In this paper we investigate non-central elements of the Iwahori-Hecke algebra of the symmetric group whose squares are central.

In particular, we describe a commutative subalgebra generated by certain non-central square roots of central elements, and the generic existence of a rank-three submodule of the Hecke algebra contained in the square root of the centre but not in the centre. The generators for this commutative subalgebra include the longest word and elements related to trivial and sign characters of the Hecke algebra. We find elegant expressions for the squares of such generators in terms of both the minimal basis of the centre and the elementary symmetric functions of Murphy elements.

Mathematics Subject Classification: 20C08.

0. INTRODUCTION

In [5], we determined explicitly, for the centre of the Hecke algebra of type A, how to express each element of the $\mathbb{Q}[q, q^{-1}]$-norm basis in [8] as a linear combination of the elements of the $\mathbb{Z}[q, q^{-1}]$-minimal basis [4]. During our research for [5], we were led naturally to the square of the element of the Hecke algebra corresponding to the longest word in the symmetric group. This square, which arises in the definition of certain norms, is well-known to be central, and to have an expression as a product of Murphy elements which follows from its analogue in the braid group. We derive directly a corresponding expression as a weighted sum of elementary symmetric functions of Murphy elements.

Motivated by recent conversations with John Murray, we have also investigated what other non-central elements of the Hecke algebra might have squares that are central. (An analogous study by Murray for the centre of the symmetric group algebra in the modular situation appears in [12].) Results of a general nature arising from our investigation can be found in Section 3, while more detailed specific cases are handled in Section 4. The non-central elements we describe in Section 3 are closely related to well-known central elements of the Hecke algebra, which are important in the study of its representation theory. Together with the

longest word, these elements span a submodule contained inside the square root of the centre, and as with the element of the Hecke algebra corresponding to the longest word of the symmetric group, the squares of these elements can be expressed as linear combinations of elementary symmetric functions of Murphy operators together with the identity (Theorem 3.8).

1. DEFINITIONS

Let $S = \{s_1, \ldots, s_{n-1}\}$ be the standard generators of the symmetric group S_n, meaning that $s_i = (i\ i+1)$ as a permutation. We say an expression $w = s_{i_1} s_{i_2} \ldots s_{i_k} \in S_n$, $s_{i_j} \in S$, is *reduced* if k is minimal. In this case, we say the length of w, denoted by $l(w)$, is k. A *partition* of n is a composition whose components are weakly decreasing from left to right. If λ is a partition of n we write $\lambda \vdash n$. The conjugacy classes in S_n are indexed by partitions of n.

There are two fairly standard and closely related sets of generators and relations with which to define the Hecke algebra $\mathcal{H} := \mathcal{H}_n$ of S_n. Let q be an indeterminate and let $R = \mathbb{Z}[q^{1/2}, q^{-1/2}]$. Then $\mathcal{H}$ is the associative R-algebra generated by the set $\{T_s \mid s \in S\}$ with identity T_1 and subject to the relations:

$$(1.1) \qquad T_s^2 = qT_1 + (q-1)T_s, \qquad \text{for } s \in S$$

$$(1.2) \qquad T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}}, \qquad \text{for } 1 \leq i \leq n-2$$

$$(1.3) \qquad T_{s_i} T_{s_j} = T_{s_j} T_{s_i} \qquad \text{for } |i-j| \geq 2.$$

The algebra $\mathcal{H}$ is a free R-module with basis $\{T_w := T_{s_{i_1}} \ldots T_{s_{i_r}} \mid w \in S_n\}$ where $w = s_{i_1} \ldots s_{i_r}$ is a reduced expression for w in S_n. We present most of the results in this paper in terms of this set of generators and relations, and with respect to this basis of $\mathcal{H}$ as a module. However, some results (Section 2) are more elegantly expressed in terms of a normalised generating set $\{\tilde{T}_s \mid s \in S\}$, defined by setting $\tilde{T}_s := q^{-1/2} T_s$. If we set $\xi = q^{1/2} - q^{-1/2}$ then the new generators satisfy the braid relations (1.2) and (1.3), but the order relation (1.1) becomes

$$(1.4) \qquad \tilde{T}_s^2 = \tilde{T}_1 + \xi \tilde{T}_s.$$

We indicate corresponding elements and substructures of this normalised version of the Hecke algebra by placing a tilde above them. For example, we denote the normalised Hecke algebra itself as $\tilde{\mathcal{H}}$. Then $\tilde{\mathcal{H}}$ is a free $\mathbb{Z}[\xi]$-module with basis $\{\tilde{T}_w := \tilde{T}_{s_{i_1}} \ldots \tilde{T}_{s_{i_r}}\}$ where $w = s_{i_1} \ldots s_{i_r}$ is a reduced expression for w in S_n.

The symmetric group S_n has a symmetry best described as the graph automorphism induced by reflecting the Dynkin diagram about its midpoint. If $\rho_n : S_n \to S_n$ is the group automorphism defined by this symmetry then $\rho_n(s_i) = s_{n-i}$ for $1 \leq i \leq n-1$. This automorphism is naturally extended to an algebra automorphism of $\mathcal{H}$, also denoted ρ_n.

The *centre* Z of $\mathcal{H}$ is defined to be the set of elements $c \in \mathcal{H}$ such that $ch = hc$ for all $h \in \mathcal{H}$. Throughout this paper we make frequent use of the 'minimal basis' for Z given in ([7, 4]). This minimal basis $\{\Gamma_\lambda \mid \lambda \vdash n\}$ is an R-basis for Z and can be characterized by the following two properties:

- $\Gamma_\lambda|_{q=1} = \sum_{w \in C_\lambda} T_w$, where C_λ is the conjugacy class of cycle type λ in S_n, and
- $\Gamma_\lambda - \sum_{w \in C_\lambda} T_w$ contains no minimal length elements from any conjugacy class.

The corresponding class elements in $\tilde{\mathcal{H}}$ are $\{\tilde{\Gamma}_\lambda \mid \lambda \vdash n\}$ with the characterizing properties $\tilde{\Gamma}_\lambda|_{\xi=0} = \sum_{w \in C_\lambda} \tilde{T}_w$, and $\tilde{\Gamma}_\lambda - \sum_{w \in C_\lambda} \tilde{T}_w$ contains no minimal length elements from any conjugacy class. We have $\tilde{\Gamma}_\lambda = q^{-l_\lambda/2} \Gamma_\lambda$, where l_λ is the length of the shortest elements in C_λ.

Write w_n for the longest word in S_n. Define $\ell := l(w_n) = \sum_{i=1}^{n-1} i$, the length of the longest word.

Define the *Poincaré polynomial* to be

$$p(q) := \sum_{w \in S_n} q^{l(w)}.$$

2. The square of the element in $\tilde{\mathcal{H}}$ corresponding to w_n

In this section we use the normalized version of the Hecke algebra $\tilde{\mathcal{H}}$ to give an explicit and surprisingly simple expression for $\tilde{T}_{w_n}^2$.

We begin with some basic facts about Murphy elements.

Definition 2.1. The *Murphy operators* or *Murphy elements* $L_{n,i}$ of $\mathcal{H}$ are defined by setting $L_{n,1} = 0$ and

$$L_{n,i} = T_{(i-1\ i)} + q^{-1}T_{(i-2\ i)} + \cdots + q^{-(i-2)}T_{(1\ i)}$$
$$= T_{s_{i-1}} + q^{-1}T_{s_{i-2}s_{i-1}s_{i-2}} + \cdots + q^{-(i-2)}T_{s_1 \ldots s_{i-1} \ldots s_1}$$

for $i = 2, 3, \ldots, n$.

By a *symmetric function in a set of commuting variables*, we mean a polynomial in those variables that is unchanged by any permutation of the variables.

Theorem 2.2 (Murphy [9] or Dipper-James [2])**.** *The Murphy elements commute. The symmetric functions in the Murphy elements are in Z.*

The following was shown by Dipper and James [2] when the Hecke algebra is defined over a suitable field, and conjectured by them to be true over R. Let $\mathcal{H}_{\mathbb{Q}(q^{1/2})}$ be the Hecke algebra taken over $\mathbb{Q}(q^{1/2})$, and let $Z_{\mathbb{Q}(q^{1/2})}$ be the centre of $\mathcal{H}_{\mathbb{Q}(q^{1/2})}$. Elements of $\mathcal{H}$ (such as the Murphy elements) are naturally embedded in $\mathcal{H}_{\mathbb{Q}(q^{1/2})}$.

Theorem 2.3 (Dipper-James [2])**.** *The set of symmetric functions in the Murphy elements in $\mathcal{H}_{\mathbb{Q}(q^{1/2})}$ is precisely $Z_{\mathbb{Q}(q^{1/2})}$.*

The Murphy elements can be similarly defined in $\tilde{\mathcal{H}}$ by setting $\tilde{L}_{n,1} := 0$ and $\tilde{L}_{n,i} := \tilde{T}_{(i-1\ i)} + \tilde{T}_{(i-2\ i)} + \cdots + \tilde{T}_{(1\ i)}$. Of course, the statement of Theorem 2.2 readily translates to the normalised context.

The i-th *elementary symmetric function* in n commuting variables is the sum of all monomials of length i in the variables whose exponents are at most 1. For example, the 2nd elementary symmetric function in the four variables x_1, x_2, x_3, x_4 is $x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4$. Define $\tilde{e}_{n,0} := \tilde{T}_1$, and for $i = 1, 2, \ldots, n-1$, let $\tilde{e}_{n,i}$ denote the i-th elementary symmetric function in the Murphy elements $\tilde{L}_{n,1}, \ldots, \tilde{L}_{n,n}$.

Proposition 2.4 ([6])**.** *For $i = 0, \ldots, n-1$,*

$$\tilde{e}_{n,i} = \sum_{l_\lambda = i} \tilde{\Gamma}_\lambda.$$

Moreover, we have the corresponding $e_{n,i} = \sum_{l_\lambda = i} \Gamma_\lambda$.

Corollary 2.5. *The set $\{\sum_{l_\lambda = i} \Gamma_\lambda \mid 1 \le i \le n-1\}$ generates $Z_{\mathbb{Q}(q^{1/2})}$ over $\mathbb{Q}(q^{1/2})$.*

Set

$$\tilde{M}_{n,i} := \rho_n(\tilde{L}_{n,i})$$
$$= \tilde{T}_{(n\ n-i+1)} + \cdots + \tilde{T}_{(n-i+2\ n-i+1)},$$

for $1 \le i \le n$. We call the $\tilde{M}_{n,i}$ the *dual Murphy elements*. These elements also appear in [13, p.26].

For example, in S_4

$$\tilde{M}_{4,1} = 0$$
$$\tilde{M}_{4,2} = \tilde{T}_{(4\ 3)} \qquad\qquad\qquad\qquad = \tilde{T}_{s_3}$$
$$\tilde{M}_{4,3} = \tilde{T}_{(4\ 2)} + \tilde{T}_{(3\ 2)} \qquad\qquad\quad = \tilde{T}_{s_2 s_3 s_2} + \tilde{T}_{s_2}$$
$$\tilde{M}_{4,4} = \tilde{T}_{(4\ 1)} + \tilde{T}_{(3\ 1)} + \tilde{T}_{(2\ 1)} \qquad = \tilde{T}_{s_1 s_2 s_3 s_2 s_1} + \tilde{T}_{s_1 s_2 s_1} + \tilde{T}_{s_1}.$$

These elements are 'dual' because if you add 'down the columns' instead of along the rows you get the standard Murphy elements. That

is, adding the first terms of each element one has $\tilde{L}_{4,4}$, adding the second terms gives $\tilde{L}_{4,3}$, and adding the third terms gives $\tilde{L}_{4,2}$.

If you like, the i-th Murphy element 'hangs down' from i to 1, while the dual version 'grows up' from $n - i + 1$ to n.

Lemma 2.6.

(1) $\sum_{i=1}^{n} \tilde{M}_{n,i} = \sum_{i=1}^{n} \tilde{L}_{n,i}$

(2) $\tilde{M}_{i,i} = \tilde{M}_{i-1,i-1} + \tilde{T}_{(1\ i)}$ for $i \geq 2$

(3) $\tilde{T}_{s_1 \ldots s_n} \tilde{T}_{s_n \ldots s_1} = \tilde{T}_1 + \xi \tilde{M}_{n+1,n+1}$

(4) For $i = 0, 1, \ldots, n - 2$, $\tilde{L}_{n+1,n+1} \tilde{e}_{n,i} + \tilde{e}_{n,i+1} = \tilde{e}_{n+1,i+1}$. For $i = n - 1$ we have $\tilde{L}_{n+1,n+1} \tilde{e}_{n,n-1} = \tilde{e}_{n+1,n}$.

(5) $\rho_n(\tilde{e}_{n,i}) = \tilde{e}_{n,i}$ for $i = 0, 1, \ldots, n - 1$.

Proof.

(1) The sums on each side are equal to $\displaystyle\sum_{1 \leq i < j \leq n} \tilde{T}_{(i\ j)}$.

(2) This follows by definition of $\tilde{M}_{i,j}$.

(3) This involves a straightforward induction on n using (2).

(4) These are elementary properties of symmetric functions in commuting variables.

(5) This also follows since $\tilde{e}_{n,i}$ is symmetric not only in the Murphy elements but also in the s_i under ρ_n.

$\square$

The following result (Eq. (2.1)) is stated in [14, §3] and [13, §6] using an alternative definition of Murphy elements in $\tilde{\mathcal{H}}$, defined by setting $\mathcal{L}_{n,1} = 1$ and $\mathcal{L}_{n,i+1} = \tilde{T}_{s_i} \mathcal{L}_{n,i} \tilde{T}_{s_i}$. It can be proved by induction on n, but is also a direct consequence of looking at the Hecke algebra as a quotient of the Braid group algebra. In the braid group, the square of the longest word is the 'full twist', and the Murphy element $\mathcal{L}_{n,i}$ is the braid whose i-th string goes behind strings $i - 1, \ldots, 1$ and returns in front of strings $1, \ldots, i-1$ to the i-th position. All other strings remain as in the identity braid. Equation (2.1) follows immediately.

$$(2.1) \qquad\qquad \tilde{T}_{w_n}^2 = \prod_{i=1}^{n} \mathcal{L}_{n,i}.$$

We present in Theorem 2.7 an alternative expression for $\tilde{T}_{w_n}^2$ which is a direct consequence of (2.1) together with the relation $\mathcal{L}_{n,i} = \xi \tilde{L}_{n,i} + \tilde{T}_1$ (the proof of this relation is elementary). We provide a direct independent proof here.

Theorem 2.7. $\tilde{T}_{w_n}^2 = \sum_{i=0}^{n-1} \xi^i \tilde{e}_{n,i}.$

Proof. The proof is by induction on n. If $n = 2$ then $\tilde{T}_{w_2}^2 = \tilde{T}_{s_1}^2 = \tilde{T}_1 + \xi \tilde{T}_{s_1} = \tilde{e}_{2,0} + \xi \tilde{e}_{2,1}$ since $\tilde{e}_{2,1} = \tilde{T}_{s_1}$.

Suppose the statement is true for n.

Now $w_{n+1} = s_1 \ldots s_n w_n = w_n s_n \ldots s_1$. Also, it is clear that $w_n s_n \ldots s_1 = s_n \ldots s_1 \rho_{n+1}(w_n)$. So, using the fact that ρ_{n+1} is an algebra automorphism,

$$
\begin{aligned}
\tilde{T}_{w_{n+1}}^2 &= \tilde{T}_{s_1 \ldots s_n} \tilde{T}_{w_n}^2 \tilde{T}_{s_n \ldots s_1} \\
&= \tilde{T}_{s_1 \ldots s_n} \tilde{T}_{s_n \ldots s_1} \rho_{n+1}\left(\tilde{T}_{w_n}^2\right) \\
&= \rho_{n+1}\left(\mathcal{L}_{n+1,n+1}\right) \rho_{n+1}\left(\sum_{i=0}^{n-1} \xi^i \tilde{e}_{n,i}\right) \quad \text{by induction} \\
&= \rho_{n+1}\left(\left(\tilde{T}_1 + \xi \tilde{L}_{n+1,n+1}\right)\left(\sum_{i=0}^{n-1} \xi^i \tilde{e}_{n,i}\right)\right) \\
&= \rho_{n+1}\big(\tilde{T}_1 + \xi(\tilde{L}_{n+1,n+1}\tilde{e}_{n,0} + \tilde{e}_{n,1}) + \xi^2(\tilde{L}_{n+1,n+1}\tilde{e}_{n,1} + \tilde{e}_{n,2}) + \ldots \\
&\qquad + \xi^{n-1}(\tilde{L}_{n+1,n+1}\tilde{e}_{n,n-2} + \tilde{e}_{n,n-1}) + \xi^n \tilde{L}_{n+1,n+1}\tilde{e}_{n,n-1}\big) \\
&= \rho_{n+1}\big(\tilde{T}_1 + \xi \tilde{e}_{n+1,1} + \xi^2 \tilde{e}_{n+1,2} + \cdots + \xi^n \tilde{e}_{n+1,n}\big) \\
&= \tilde{T}_1 + \xi \tilde{e}_{n+1,1} + \xi^2 \tilde{e}_{n+1,2} + \cdots + \xi^n \tilde{e}_{n+1,n}
\end{aligned}
$$

as desired (the last equality due to Lemma 2.6(5), the second to last due to 2.6(4)). $\qquad\square$

Corollary 2.8. *In terms of class elements,*

$$\tilde{T}_{w_n}^2 = \sum_{\lambda \vdash n} \xi^{l_\lambda} \tilde{\Gamma}_\lambda.$$

Proof. This is an immediate consequence of Theorem 2.7 and Proposition 2.4. $\qquad\square$

Lastly, we rewrite Corollary 2.8 over R.

Recall the transformations $\tilde{T}_s = q^{-1/2} T_s$ and $\xi = q^{1/2} - q^{-1/2}$. So $\tilde{T}_{w_n}^2 = (q^{-\ell/2} T_{w_n})^2 = q^{-\ell} T_{w_n}^2$. We also have $\tilde{\Gamma}_\lambda = q^{-l_\lambda/2} \Gamma_\lambda$, and so $\xi^{l_\lambda} \tilde{\Gamma}_\lambda = q^{-l_\lambda/2}(q^{1/2} - q^{-1/2})^{l_\lambda} \Gamma_\lambda = q^{-l_\lambda}(q-1)^{l_\lambda} \Gamma_\lambda$. Recalling from Proposition 2.4 that $e_{n,i} = \sum_{l_\lambda = i} \Gamma_\lambda$, the statement of Corollary 2.8 becomes

Corollary 2.9.
$$T_{w_n}^2 = q^\ell \sum_{\lambda \vdash n} \left(1 - q^{-1}\right)^{l_\lambda} \Gamma_\lambda.$$

3. The square root of the centre of $\mathcal{H}$

When $n = 2$ the algebra $\mathcal{H}_n$ is commutative, so we restrict our attention in this section to $n \geq 3$.

Definition 3.1. The *square root* of Z, denoted $\sqrt{Z}$, is defined to be:
$$\sqrt{Z} := \{h \in \mathcal{H} \mid h^2 \in Z\}.$$

Clearly $Z \subseteq \sqrt{Z}$. Also, $Z \neq \sqrt{Z}$ since $T_{w_n} \in \sqrt{Z} \setminus Z$. The fact that $T_{w_n} \in \sqrt{Z} \setminus Z$ is well-known and follows from the centrality of the square of the corresponding longest element in the braid group (of which $\mathcal{H}$ is a quotient), where that square is the full twist. In this section we define two additional elements which we show are in $\sqrt{Z}$, and together with T_{w_n}, generate a commutative subalgebra of $\mathcal{H}$. In Theorem 3.8 we give the forms of the squares of these elements.

We say an element $v \in \mathcal{H}$ is a right *eigenvector* for multiplication by h if $hv = kv$ for some $k \in R$. Left eigenvectors are defined similarly.

Recall the following well-known elements of Z:
$$x := \sum_{w \in S_n} T_w$$
$$y := \sum_{w \in S_n} (-q)^{\ell - l(w)} T_w.$$

These elements were defined in [1, Section 3] and used there to study the permutation modules of $\mathcal{H}$. They are widely used in the representation theory of $\mathcal{H}$ (for example in [2], [10], [11], and [3] among many other places). The modules $x\mathcal{H}$ and $y\mathcal{H}$ are called the trivial and alternating $\mathcal{H}$-modules respectively. Analogous elements can be defined for parabolic subalgebras of $\mathcal{H}$. Both x and y are eigenvectors for the action of the generators of $\mathcal{H}$ (Lemma 3.2 (1)), a property which they share with some other elements of $\sqrt{Z}$ (see Section 4).

The following result is well-known.

Lemma 3.2.
(1) *For any $s \in S$, $T_s x = x T_s = qx$ and $T_s y = y T_s = -y$. Moreover, x and y are central.*
(2) $xy = 0$.
(3) $x = \sum_{\lambda \vdash n} \Gamma_\lambda$ and $y = \sum_{\lambda \vdash n} (-q)^{\ell - l_\lambda} \Gamma_\lambda$.
(4) $x^2 = p(q)x$ and $y^2 = (-1)^\ell p(q)y$.

We now define the elements $\bar{x}$ and $\bar{y}$, and investigate their properties in the remainder of this paper:

$$\bar{x} := x - T_{w_n}$$
$$\bar{y} := y - T_{w_n}.$$

Lemma 3.3. *The following hold:*

(1) $\bar{x}T_{w_n} = T_{w_n}\bar{x}$,
(2) $\bar{y}T_{w_n} = T_{w_n}\bar{y}$,
(3) $\bar{x}T_{w_n}, \bar{y}T_{w_n}, \bar{x}\bar{y} \in Z$,
(4) $\bar{x}\bar{y} = \bar{y}\bar{x}$.

Proof. The centrality of x and y from Lemma 3.2(1) implies (1) and (2). Now w_n acts on the set S of generators of S_n via the graph automorphism ρ_n defined in Section 1, and this action extends to the Hecke algebra level as follows (remembering that $l(sw_n) = l(w_ns) = l(w_n)-1$ for any $s \in S$): $T_{w_n}T_s = T_{\rho_n(s)}T_{\rho_n(s)w_n}T_s = T_{\rho_n(s)}T_{w_ns}T_s = T_{\rho_n(s)}T_{w_n}$. Since $T_sx = xT_{s'}$ for any $s, s' \in S$, we have in particular $T_sx = xT_{\rho_n(s)}$. Hence, for any $s \in S$ (using the fact that ρ_n is an involution), we have

$$\bar{x}T_{w_n}T_s = \bar{x}T_{\rho_n(s)}T_{w_n}$$
$$= (x - T_{w_n})T_{\rho_n(s)}T_{w_n}$$
$$= \left(xT_{\rho_n(s)} - T_{w_n}T_{\rho_n(s)}\right)T_{w_n}$$
$$= (T_sx - T_sT_{w_n})T_{w_n}$$
$$= T_s(x - T_{w_n})T_{w_n}$$
$$= T_s\bar{x}T_{w_n}.$$

Therefore, $\bar{x}T_{w_n} \in Z$. Similar arguments show that $\bar{y}T_{w_n}$ and $\bar{x}\bar{y}$ are central, proving (3). Finally, the centrality of x implies $\bar{y}x = x\bar{y}$, which together with (2) implies (4), and the proof is complete. $\square$

Proposition 3.4. *The elements $\bar{x}$, $\bar{y}$, T_{w_n} are in $\sqrt{Z} \setminus Z$.*

Proof. As already mentioned, it is well known that $T_{w_n} \in \sqrt{Z} \setminus Z$. By Lemma 3.2(1), we have that $x, y \in Z$, from which we deduce that $\bar{x}, \bar{y} \notin Z$. Since $x^2, y^2, T_{w_n}^2 \in Z$, and since T_{w_n} commutes with $\bar{x}$ and $\bar{y}$ (Lemma 3.3), it follows that $\bar{x}^2, \bar{y}^2 \in Z$. $\square$

Remark. Note that while $\bar{x} + \bar{y} \notin Z$, some linear combinations of $\bar{x}$ and $\bar{y}$ are in Z (for instance $\bar{x} - \bar{y}$), and all linear combinations of $\bar{x}$ and $\bar{y}$ are in $\sqrt{Z}$ since $\bar{x}^2, \bar{y}^2, \bar{x}\bar{y} \in Z$.

The following is an immediate consequence of Lemma 3.3 and Proposition 3.4.

Corollary 3.5. *The R-submodule of $\mathcal{H}$ spanned by $\{\bar{x}, \bar{y}, T_{w_n}\}$ is contained in $\sqrt{Z} \setminus Z$.*

Proposition 3.6. *The elements $\bar{x}$ and $\bar{y}$ are not zero divisors.*

Proof. Suppose that $\bar{x}h = (x - T_{w_n})h = 0$ for some $h \in \mathcal{H}$. Multiplying on the left by x gives $(x^2 - xT_{w_n})h = 0$, and therefore, using Lemma 3.2 parts 1 and 4, we have $\left(p(q) - q^{l(w_n)}\right)xh = 0$. This gives $xh = 0$ since $\mathcal{H}$ is a free R-module, and subtracting $xh = 0$ from $(x - T_{w_n})h = 0$ we obtain $T_{w_n}h = 0$. Thus $h = 0$ since T_w is invertible for all $w \in S_n$. The proof for $\bar{y}$ is similar. $\square$

Proposition 3.7. *Let $\mathcal{X}$ be the R-subalgebra of $\mathcal{H}$ generated by $\{\bar{x}, \bar{y}, T_{w_n}\}$. Then $\mathcal{X}$ is commutative.*

Proof. This is immediate from Lemma 3.3. $\square$

Remark. Since $\bar{x}\bar{y}T_{w_n} \notin Z$, we have that $\bar{x} + \bar{y}T_{w_n} \notin \sqrt{Z}$. Hence, $\mathcal{X} \not\subseteq \sqrt{Z}$. This also shows that $\sqrt{Z}$ itself is not a subalgebra of $\mathcal{H}$. However, $\{\bar{x}^a\bar{y}^bT_{w_n}^c \mid a, b, c \in \mathbb{N}\} \subseteq \sqrt{Z}$, and, in fact, all elements of $\mathcal{X}$ with even exponent sum are central.

Unfortunately, $\sqrt{Z}$ (which includes Z) is not even an R-module. Although we have shown that all R-linear combinations of the particular elements $\bar{x}$, $\bar{y}$ and T_{w_n} are contained in $\sqrt{Z}$ (Corollary 3.5), it is not true in general that other elements of $\sqrt{Z}$ have this property (see Section 4), and certainly unlikely that the sum of a non-central element in $\sqrt{Z}$ and an element of Z will still be in $\sqrt{Z}$.

Remark. There are no subalgebras of $\mathcal{H}$ which are contained in $\sqrt{Z}$ that contain elements in $\sqrt{Z} \setminus Z$. This is because, for instance, if $h \in \sqrt{Z} \setminus Z$, then $h + h^2 \notin \sqrt{Z}$.

Theorem 3.8. *The squares of $\bar{x}$ and $\bar{y}$ can be expressed as linear combinations of elementary symmetric functions of Murphy elements together with the identity. Specifically,*

$$\bar{x}^2 = \sum_{\lambda \vdash n} \left(p(q) - 2q^\ell + q^{\ell - l_\lambda}(q-1)^{l_\lambda}\right)\Gamma_\lambda$$

$$= \sum_{i=0}^{n-1} \left(p(q) - 2q^\ell + q^{\ell - i}(q-1)^i\right)e_{n,i}, \text{ and}$$

$$\bar{y}^2 = \sum_{\lambda \vdash n} (-1)^{l_\lambda} q^{\ell - l_\lambda}\left(p(q) - 2 + (1-q)^{l_\lambda}\right)\Gamma_\lambda$$

$$= \sum_{i=0}^{n-1} (-1)^i q^{\ell - i}\left(p(q) - 2 + (1-q)^i\right)e_{n,i}.$$

Proof. Immediately from the definition of $\bar{x}$ and $\bar{y}$ we have that

$$\bar{x}^2 = x^2 - 2xT_{w_n} + T_{w_n}^2, \text{ and}$$
$$\bar{y}^2 = y^2 - 2yT_{w_n} + T_{w_n}^2.$$

The forms for x^2, y^2 and $T_{w_n}^2$ from Lemma 3.2 parts 3 and 4 and Corollary 2.9, together with Lemma 3.2 part 1, then give that

$$\bar{x}^2 = p(q)x - 2q^\ell x + T_{w_n}^2$$
$$= \left(p(q) - 2q^\ell\right)\sum_{\lambda \vdash n}\Gamma_\lambda + q^\ell\sum_{\lambda \vdash n}\left(1 - q^{-1}\right)^{l_\lambda}\Gamma_\lambda.$$

The expression for $\bar{x}^2$ in terms of $\{\Gamma_\lambda \mid \lambda \vdash n\}$ follows. The corresponding expression for $\bar{y}^2$ is similarly derived. The expressions for both $\bar{x}^2$ and $\bar{y}^2$ in terms of $\{e_{n,i} \mid 0 \le i \le n - 1\}$ follow immediately from Proposition 2.4. $\qquad\square$

4. EXAMPLES: $\sqrt{Z}$ FOR $\mathcal{H}_3$ AND $\mathcal{H}_4$

We have seen in Section 3 that the commutative subalgebra $\mathcal{X}$ of $\mathcal{H}$ generated by $\{\bar{x}, \bar{y}, T_{w_n}\}$ is not contained in $\sqrt{Z}$. We see in the following examples that it is also the case that $\sqrt{Z}$ is not contained in $\mathcal{X}$.

4.1. $\mathcal{H}_3$.
When $n = 3$ it is easy to check that while the elements $T_{s_1} - T_{s_2}$ and $T_{s_1 s_2} - T_{s_2 s_1}$ are in $\sqrt{Z}$, their span is not in $\sqrt{Z}$ — unlike $\bar{x}_3$, $\bar{y}_3$ and $T_{w_3} = T_{s_1 s_2 s_1}$. However they do have the interesting property that they are eigenvectors for the multiplicative action of Z. This parallels a similar fact for x and y (Lemma 3.2(1)). Recall from Section 1 that $\{\Gamma_{(1,1,1)} = T_1, \Gamma_{(2,1)} = T_{s_1} + T_{s_2} + q^{-1}T_{s_1 s_2 s_1}, \Gamma_{(3)} = T_{s_1 s_2} + T_{s_2 s_1} + q^{-1}(q - 1)T_{s_1 s_2 s_1}\}$ is an R-basis for Z. Then we have

$$\begin{aligned}
\Gamma_{(2,1)}(T_1 - T_2) &= (q - 1)(T_1 - T_2)\\
\Gamma_{(3)}(T_1 - T_2) &= -q(T_1 - T_2)\\
\Gamma_{(2,1)}(T_{12} - T_{21}) &= (q - 1)(T_{12} - T_{21})\\
\Gamma_{(3)}(T_{12} - T_{21}) &= -q(T_{12} - T_{21}).
\end{aligned}$$

Proposition 4.1. *When $n = 3$, every element in $\sqrt{Z}\backslash Z$ can be written as a $\mathbb{Q}(q)$–linear combination of the following elements:*

$$\bar{x}_3 := T_1 + T_{s_1} + T_{s_2} + T_{s_1 s_2} + T_{s_2 s_1}$$
$$\bar{y}_3 := T_1 - q^{-1}(T_{s_1} + T_{s_2}) + q^{-2}(T_{s_1 s_2} + T_{s_2 s_1})$$
$$T_{w_3} := T_{s_1 s_2 s_1}$$
$$R_4 := T_{s_1} - T_{s_2}$$
$$R_5 := T_{s_1 s_2} - T_{s_2 s_1}.$$

Proof. Solving the system of equations created by evaluating

$$(a_1 T_1 + a_2 T_{s_1} + a_3 T_{s_2} + a_4 T_{s_1 s_2} + a_5 T_{s_2 s_1} + a_6 T_{s_1 s_2 s_1})^2 T_s$$
$$= T_s(a_1 T_1 + a_2 T_{s_1} + a_3 T_{s_2} + a_4 T_{s_1 s_2} + a_5 T_{s_2 s_1} + a_6 T_{s_1 s_2 s_1})^2$$

for each $s \in S = \{s_1, s_2\}$ yields the following two solutions:

(4.1) $\qquad \{a_2 = a_3, \ a_4 = a_5, \ a_6 = q^{-1}a_3 + q^{-1}(q-1)a_5\}$

and

(4.2) $\qquad \{a_1 = -\dfrac{q-1}{2}(a_2 + a_3) + \dfrac{q}{2}(a_4 + a_5)\}.$

The first solution, (4.1), characterizes the elements of Z: the relations given are exactly those given by [4, Lemma (3.1)]. Thus the elements of $\sqrt{Z}\backslash Z$ must satisfy equations (4.2). The elements $\{\bar{x}_3, \bar{y}_3, T_{w_3}, R_4, R_5\}$ all satisfy these equations, and are easily seen to be linearly independent. Thus they span a five-dimensional subspace of the solution space, which itself has five dimensions (there are five free parameters). Hence they span the solution space of (4.2). $\qquad \square$

Corollary 4.2. *For any subset W of $\mathcal{H}$, let $\langle W \rangle$ denote the R-span of W. Then $\langle \{\bar{x}_3, \bar{y}_3, T_{w_3}, R_4, R_5\} \rangle \cap Z_3 = \langle \{x_3, y_3\} \rangle.$*

Proof. This is immediate from common solutions to (4.1) and (4.2). $\qquad \square$

The squares of these five square roots of central elements (using Theorems 3.8 and 2.7) are:

$$\bar{x}_3^2 = (2q^2 + 2q + 1)\Gamma_{(1,1,1)} + (q+1)^2\Gamma_{(2,1)} + (3q+1)\Gamma_{(3)},$$
$$\bar{y}_3^2 = q^4(q^2 + 2q + 2)\Gamma_{(1,1,1)} - q^3(q+1)^2\Gamma_{(2,1)} + q^3(q+3)\Gamma_{(3)},$$
$$T_{w_3}^2 = \Gamma_{(1,1,1)} + (1 - q^{-1})\Gamma_{(2,1)} + (1 - q^{-1})^2\Gamma_{(3)}$$
$$R_4^2 = 2q\Gamma_{(1,1,1)} + (q-1)\Gamma_{(2,1)} - \Gamma_{(3)}$$
$$R_5^2 = -2q^2\Gamma_{(1,1,1)} - q(q-1)\Gamma_{(2,1)} + q\Gamma_{(3)}.$$

Interestingly, we have the relation $R_5^2 = -qR_4^2$. It follows that the central element given by R_4^2 has at least four distinct square roots over $\mathbb{C}(q^{1/2})$.

4.2. $\mathcal{H}_4$.

As with $\mathcal{H}_3$, the algebra $\mathcal{X}$ generated by $\{\bar{x}_4, \bar{y}_4, T_{w_4}\}$ does not contain all elements of $\sqrt{Z} \setminus Z$. Motivated by the relationships among the elements $\bar{x}_4$, $\bar{y}_4$ and the eigenvectors of Z when $n = 4$, as well as the experience in $\mathcal{H}_3$, we have identified several more elements of $\sqrt{Z}$ by looking for eigenvectors of the multiplicative action of class elements. The following are additional examples of square roots, which are clearly non-central since the coefficients of the shortest elements of conjugacy classes are not equal:

$$
\begin{aligned}
R_4 := {} & q\,(T_{s_1 s_2} - T_{s_2 s_1} + T_{s_3 s_2} - T_{s_2 s_3}) + (q-1)(T_{s_1 s_3 s_2} - T_{s_2 s_1 s_3}) \\
& + T_{s_1 s_2 s_1 s_3} - T_{s_1 s_2 s_3 s_2} + T_{s_2 s_3 s_2 s_1} - T_{s_1 s_3 s_2 s_1}. \\
R_5 := {} & q^2(T_{s_1} + T_{s_3}) + q(q-1)(T_{s_2 s_1} + T_{s_2 s_3} + T_{s_1 s_3}) + (q-1)^2 T_{s_2 s_1 s_3} \\
& - q(T_{s_1 s_2 s_1} + T_{s_2 s_3 s_2} + T_{s_1 s_2 s_3} + T_{s_3 s_2 s_1}) - (q-1)(T_{s_1 s_2 s_1 s_3} \\
& + T_{s_2 s_3 s_2 s_1} + T_{s_2 s_1 s_3 s_2}) + T_{s_1 s_2 s_1 s_3 s_2} + T_{s_2 s_1 s_3 s_2 s_1}. \\
R_6 := {} & q^2 T_{s_2} + q(q-1)(T_{s_1 s_2} + T_{s_3 s_2}) - q(T_{s_1 s_2 s_1} + T_{s_2 s_3 s_2} + T_{s_1 s_2 s_3} \\
& + T_{s_3 s_2 s_1} - T_{s_2 s_1 s_3}) + (q^2 - q + 1)T_{s_1 s_3 s_2} - (q-1)(T_{s_1 s_2 s_3 s_2} \\
& + T_{s_1 s_3 s_2 s_1}) + T_{s_1 s_2 s_3 s_2 s_1}.
\end{aligned}
$$

The set $\{\bar{x}_4, \bar{y}_4, T_{w_4}, R_4, R_5, R_6\}$ is linearly independent. While the elements $\{\bar{x}_4, \bar{y}_4, T_{w_4}\}$ commute with each other and with elements of $\{R_4, R_5, R_6\}$, the elements of $\{R_4, R_5, R_6\}$ do not commute among themselves.

Unlike the $\mathcal{H}_3$ case, we cannot use a dimension argument here to prove that we have all square roots. In fact, as the dimension of the algebra here is 24, it is rather unlikely that this set spans all square roots.

REFERENCES

[1] Richard Dipper and Gordon James, 'Representations of Hecke algebras of general linear groups', *Proc. London Math. Soc.* (3) **52** (1986), 20–52.

[2] ———, 'Blocks and idempotents of Hecke algebras of general linear groups', *Proc. London Math. Soc. (3)* (1) **54** (1987), 57–82.

[3] Karin Erdmann and Daniel K. Nakano, 'Representation type of Hecke algebras of type A', *Trans. Amer. Math. Soc.* (1) **354** (2002), 275–285 (electronic).

[4] Andrew Francis, 'The minimal basis for the centre of an Iwahori-Hecke algebra', *J. Algebra* (1) **221** (1999), 1–28.

[5] Andrew Francis and Lenny Jones, 'On bases of centres of Iwahori-Hecke algebras of the symmetric group', *J. Algebra* **289** (2005), 42–69.

[6] Andrew R. Francis and John J. Graham, 'Centres of Hecke algebras: the Dipper-James conjecture', *J. Algebra* **306** (2006) 244267.

[7] Meinolf Geck and Raphaël Rouquier, 'Centers and simple modules for Iwahori-Hecke algebras', in: *Finite reductive groups (Luminy, 1994)* (Birkhäuser Boston, Boston, MA, 1997) pp. 251–272.

[8] Lenny K. Jones, 'Centers of generic Hecke algebras', *Trans. Amer. Math. Soc.* (1) **317** (1990), 361–392.

[9] G. E. Murphy, 'The idempotents of the symmetric group and Nakayama's conjecture', *J.Algebra* **81** (1983), 258–265.

[10] ——, 'On the representation theory of the symmetric groups and associated Hecke algebras', *J. Algebra* (2) **152** (1992), 492–513.

[11] ——, 'The representations of Hecke algebras of type A_n', *J. Algebra* (1) **173** (1995), 97–121.

[12] John Murray, 'Squares in the centre of the group algebra of a symmetric group', *Bull. London Math. Soc.* **34** (2002), 155–164.

[13] O. Ogievetsky and P. Pyatov, 'Lecture on Hecke algebras', unpublished, 1999.

[14] Arun Ram, 'Seminormal representations of Weyl groups and Iwahori-Hecke algebras', *Proc. London Math. Soc.* (3) **75** (1997), 99–133.

SCHOOL OF COMPUTING AND MATHEMATICS,, UNIVERSITY OF WESTERN SYDNEY,, NSW 1797, AUSTRALIA

DEPARTMENT OF MATHEMATICS,, SHIPPENSBURG UNIVERSITY,, PENNSYLVANIA, USA